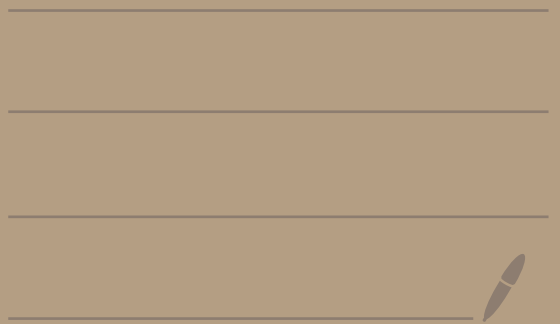


2550

HW 7

Part 1

Solutions



① (a)

$$\text{span}(\{\langle 0, 1 \rangle, \langle 1, 1 \rangle, \langle -3, 2 \rangle\})$$

$$= \left\{ c_1 \langle 0, 1 \rangle + c_2 \langle 1, 1 \rangle + c_3 \langle -3, 2 \rangle \mid \left. \begin{array}{l} c_1, c_2, c_3 \\ \text{are real} \\ \text{numbers} \end{array} \right\} \right.$$

Five example vectors in the above span:

$$1 \cdot \langle 0, 1 \rangle + 1 \cdot \langle 1, 1 \rangle + 1 \cdot \langle -3, 2 \rangle = \langle -2, 4 \rangle$$

$$0 \cdot \langle 0, 1 \rangle + 0 \cdot \langle 1, 1 \rangle + 0 \cdot \langle -3, 2 \rangle = \langle 0, 0 \rangle$$

$$10 \cdot \langle 0, 1 \rangle + \pi \langle 1, 1 \rangle + 0 \cdot \langle -3, 2 \rangle = \langle \pi, \pi + 10 \rangle$$

$$0 \cdot \langle 0, 1 \rangle + 1 \cdot \langle 1, 1 \rangle - 5 \cdot \langle -3, 2 \rangle = \langle 16, -9 \rangle$$

$$2 \cdot \langle 0, 1 \rangle - 1 \cdot \langle 1, 1 \rangle + 1 \cdot \langle -3, 2 \rangle = \langle -4, 3 \rangle$$

①(b)

$$\text{span}(\{ \langle 0, -2, 2 \rangle, \langle 1, 3, -1 \rangle \}) \\ = \left\{ c_1 \langle 0, -2, 2 \rangle + c_2 \langle 1, 3, -1 \rangle \mid c_1, c_2 \in \mathbb{R} \right\}$$

Five example vectors in the above span:

$$0 \cdot \langle 0, -2, 2 \rangle + 0 \cdot \langle 1, 3, -1 \rangle = \langle 0, 0, 0 \rangle$$

$$1 \cdot \langle 0, -2, 2 \rangle + 0 \cdot \langle 1, 3, -1 \rangle = \langle 0, -2, 2 \rangle$$

$$0 \cdot \langle 0, -2, 2 \rangle + 1 \cdot \langle 1, 3, -1 \rangle = \langle 1, 3, -1 \rangle$$

$$\frac{1}{2} \cdot \langle 0, -2, 2 \rangle - 2 \langle 1, 3, -1 \rangle = \langle -2, -7, 3 \rangle$$

$$5 \cdot \langle 0, -2, 2 \rangle + 1 \cdot \langle 1, 3, -1 \rangle = \langle 1, -7, 9 \rangle$$

①(c)

$$\text{span}(\{2, 1+x\})$$

$$= \left\{ c_1 \cdot 2 + c_2 \cdot (1+x) \mid c_1, c_2 \text{ are real numbers} \right\}$$

Five example vectors in the above span:

$$0 \cdot 2 + 0 \cdot (1+x) = 0$$

$$1 \cdot 2 + 0 \cdot (1+x) = 2$$

$$0 \cdot 2 + \frac{1}{2} \cdot (1+x) = \frac{1}{2} + \frac{1}{2}x$$

$$\left(-\frac{1}{2}\right) \cdot 2 + \pi \cdot (1+x) = (-1+\pi) + \pi x$$

$$10 \cdot 2 - 10 \cdot (1+x) = 10 - 10x$$

①(d)

$$\text{span}\left(\{-1-2x, x^2, 1+x+x^2\}\right)$$

$$= \left\{ c_1(-1-2x) + c_2(x^2) + c_3(1+x+x^2) \mid c_1, c_2, c_3 \in \mathbb{R} \right\}$$

Five example vectors in the above span:

$$1 \cdot (-1-2x) + 0 \cdot x^2 + 0 \cdot (1+x+x^2) = -1-2x$$

$$-1 \cdot (-1-2x) + 1 \cdot x^2 - 1 \cdot (1+x+x^2) = x$$

$$2 \cdot (-1-2x) - 2 \cdot x^2 + 1 \cdot (1+x+x^2) = -1-3x-x^2$$

$$4 \cdot (-1-2x) + 0 \cdot x^2 + 0 \cdot (1+x+x^2) = -4-8x$$

$$0 \cdot (-1-2x) + 5 \cdot x^2 - 5 \cdot (1+x+x^2) = -5-5x$$

②(a)

We want to know if we can write

$$\langle 2, 2, 2 \rangle = c_1 \langle 0, -2, 2 \rangle + c_2 \langle 1, 3, -1 \rangle$$

$\underbrace{\hspace{15em}}_{c_1 \vec{u} + c_2 \vec{v}}$

Let's see.

The above equation becomes:

$$\langle 2, 2, 2 \rangle = \langle c_2, -2c_1 + 3c_2, 2c_1 - c_2 \rangle$$

Now equate each component to get

$$\begin{cases} 2 = c_2 \\ 2 = -2c_1 + 3c_2 \\ 2 = 2c_1 - c_2 \end{cases}$$

Let's see if we can solve this system

$$\begin{aligned} &\downarrow \\ &\begin{pmatrix} 0 & 1 & | & 2 \\ -2 & 3 & | & 2 \\ 2 & -1 & | & 2 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{pmatrix} 2 & -1 & | & 2 \\ -2 & 3 & | & 2 \\ 0 & 1 & | & 2 \end{pmatrix} \xrightarrow{\frac{1}{2}R_1 \rightarrow R_1} \begin{pmatrix} 1 & -\frac{1}{2} & | & 1 \\ -2 & 3 & | & 2 \\ 0 & 1 & | & 2 \end{pmatrix} \\ &\xrightarrow{2R_1 + R_2 \rightarrow R_2} \begin{pmatrix} 1 & -\frac{1}{2} & | & 1 \\ 0 & 2 & | & 4 \\ 0 & 1 & | & 2 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & -\frac{1}{2} & | & 1 \\ 0 & 1 & | & 2 \\ 0 & 2 & | & 4 \end{pmatrix} \rightarrow \end{aligned}$$

$$\xrightarrow{-2R_2 + R_3 \rightarrow R_3} \left( \begin{array}{cc|c} 1 & -1/2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array} \right)$$

So we have

$$\begin{array}{l} c_1 - \frac{1}{2}c_2 = 1 \quad \textcircled{1} \\ c_2 = 2 \quad \textcircled{2} \\ 0 = 0 \end{array}$$

$$\textcircled{2} \text{ gives } c_2 = 2$$

$$\textcircled{1} \text{ gives } c_1 = 1 + \frac{1}{2}c_2 \\ = 1 + \frac{1}{2}(2) = 2$$

Thus,

$$\begin{aligned} \langle 2, 2, 2 \rangle &= 2 \cdot \langle 0, -2, 2 \rangle + 2 \cdot \langle 1, 3, -1 \rangle \\ &= 2 \cdot \vec{u} + 2 \vec{v} \end{aligned}$$

So,

$\langle 2, 2, 2 \rangle$  is in the span of  $\vec{u}$  and  $\vec{v}$ .

②(b)

We want to know if we can write

$$\langle 3, 1, 5 \rangle = c_1 \langle 0, -2, 2 \rangle + c_2 \langle 1, 3, -1 \rangle$$

Let's see.

The above equation becomes:

$$\langle 3, 1, 5 \rangle = \langle c_2, -2c_1 + 3c_2, 2c_1 - c_2 \rangle$$

Now equate each component to get

$$\begin{cases} 3 = c_2 \\ 1 = -2c_1 + 3c_2 \\ 5 = 2c_1 - c_2 \end{cases}$$

Let's see if we can solve this system

$$\begin{aligned} &\begin{pmatrix} 0 & 1 & | & 3 \\ -2 & 3 & | & 1 \\ 2 & -1 & | & 5 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{pmatrix} 2 & -1 & | & 5 \\ -2 & 3 & | & 1 \\ 0 & 1 & | & 3 \end{pmatrix} \xrightarrow{\frac{1}{2}R_1 \rightarrow R_1} \begin{pmatrix} 1 & -\frac{1}{2} & | & \frac{5}{2} \\ -2 & 3 & | & 1 \\ 0 & 1 & | & 3 \end{pmatrix} \\ &\xrightarrow{2R_1 + R_2 \rightarrow R_2} \begin{pmatrix} 1 & -\frac{1}{2} & | & \frac{5}{2} \\ 0 & 2 & | & 6 \\ 0 & 1 & | & 3 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & -\frac{1}{2} & | & \frac{5}{2} \\ 0 & 1 & | & 3 \\ 0 & 2 & | & 6 \end{pmatrix} \rightarrow \end{aligned}$$



$$\xrightarrow{-2R_2 + R_3 \rightarrow R_3} \left( \begin{array}{cc|c} 1 & -1/2 & 5/2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array} \right)$$

So we have

$$\begin{array}{l} c_1 - \frac{1}{2} c_2 = 5/2 \quad \textcircled{1} \\ c_2 = 3 \quad \textcircled{2} \\ 0 = 0 \end{array}$$

$$\textcircled{2} \text{ gives } c_2 = 3$$

$$\begin{aligned} \textcircled{1} \text{ gives } c_1 &= 5/2 + \frac{1}{2} c_2 \\ &= 5/2 + \frac{1}{2} (3) \\ &= 8/2 = 4 \end{aligned}$$

Thus,

$$\langle 3, 1, 5 \rangle = 4 \langle 0, -2, 2 \rangle + 3 \langle 1, 3, -1 \rangle$$

So,

$\langle 3, 1, 5 \rangle$  is in the span of  $\vec{u}$  and  $\vec{v}$ .

②(c)

We want to know if we can write

$$\langle 0, 4, 5 \rangle = c_1 \langle 0, -2, 2 \rangle + c_2 \langle 1, 3, -1 \rangle$$

Let's see.

The above equation becomes:

$$\langle 0, 4, 5 \rangle = \langle c_2, -2c_1 + 3c_2, 2c_1 - c_2 \rangle$$

Now equate each component to get

$$\begin{cases} 0 = c_2 \\ 4 = -2c_1 + 3c_2 \\ 5 = 2c_1 - c_2 \end{cases}$$

Let's see if we can solve this system

$$\begin{aligned} &\downarrow \\ &\begin{pmatrix} 0 & 1 & | & 0 \\ -2 & 3 & | & 4 \\ 2 & -1 & | & 5 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{pmatrix} 2 & -1 & | & 5 \\ -2 & 3 & | & 4 \\ 0 & 1 & | & 0 \end{pmatrix} \xrightarrow{\frac{1}{2}R_1 \rightarrow R_1} \begin{pmatrix} 1 & -\frac{1}{2} & | & \frac{5}{2} \\ -2 & 3 & | & 4 \\ 0 & 1 & | & 0 \end{pmatrix} \\ &\xrightarrow{2R_1 + R_2 \rightarrow R_2} \begin{pmatrix} 1 & -\frac{1}{2} & | & \frac{5}{2} \\ 0 & 2 & | & 9 \\ 0 & 1 & | & 0 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & -\frac{1}{2} & | & \frac{5}{2} \\ 0 & 1 & | & 0 \\ 0 & 2 & | & 9 \end{pmatrix} \rightarrow \end{aligned}$$

$$\xrightarrow{-2R_2 + R_3 \rightarrow R_3} \left( \begin{array}{cc|c} 1 & -1/2 & 5/2 \\ 0 & 1 & 0 \\ 0 & 0 & 9 \end{array} \right)$$

So we have

$$c_1 - \frac{1}{2}c_2 = 5/2 \quad \textcircled{1}$$

$$c_2 = 0 \quad \textcircled{2}$$

$$0 = 9 \quad \textcircled{3}$$

← This last equation  $0=9$  shows that the system has no solutions

Thus, there is no solution to

$$\begin{aligned} \langle 0, 4, 5 \rangle &= c_1 \cdot \langle 0, -2, 2 \rangle + c_2 \cdot \langle 1, 3, -1 \rangle \\ &= c_1 \vec{u} + c_2 \vec{v} \end{aligned}$$

$\langle 0, 4, 5 \rangle$  is not in the span of  $\vec{u}$  and  $\vec{v}$ .

(2)(d) You can proceed as in the previous problems, but this one is easy to solve.

We have

$$\begin{aligned}\langle 0, 0, 0 \rangle &= 0 \cdot \langle 0, -2, 2 \rangle + 0 \cdot \langle 1, 3, -1 \rangle \\ &= 0 \cdot \vec{u} + 0 \cdot \vec{v}\end{aligned}$$

Thus,  $\langle 0, 0, 0 \rangle$  is in the span of  $\vec{u}$  and  $\vec{v}$ .

③ (a)

We want to know if we can write

$$\begin{aligned} 3 + 2x + x^2 + 2x^3 &= c_1 \vec{p}_1 + c_2 \vec{p}_2 + c_3 \vec{p}_3 \\ &= c_1(2 + x + 4x^2) + c_2(1 - x + 3x^2) + c_3(1 + x^3) \end{aligned}$$

This simplifies to

$$3 + 2x + x^2 + 2x^3 = (2c_1 + c_2 + c_3) + (c_1 - c_2)x + (4c_1 + 3c_2)x^2 + c_3x^3$$

Now equate the coefficients of both sides to get:

$$\begin{aligned} 3 &= 2c_1 + c_2 + c_3 \\ 2 &= c_1 - c_2 \\ 1 &= 4c_1 + 3c_2 \\ 2 &= c_3 \end{aligned}$$

Now we must see if this system has a solution or not

$$\begin{pmatrix} 2 & 1 & 1 & | & 3 \\ 1 & -1 & 0 & | & 2 \\ 4 & 3 & 0 & | & 1 \\ 0 & 0 & 1 & | & 2 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & -1 & 0 & | & 2 \\ 2 & 1 & 1 & | & 3 \\ 4 & 3 & 0 & | & 1 \\ 0 & 0 & 1 & | & 2 \end{pmatrix}$$

$$\begin{array}{l} -2R_1 + R_2 \rightarrow R_2 \\ -4R_1 + R_3 \rightarrow R_3 \end{array} \xrightarrow{\hspace{1cm}} \begin{pmatrix} 1 & -1 & 0 & | & 2 \\ 0 & 3 & 1 & | & -1 \\ 0 & 7 & 0 & | & -7 \\ 0 & 0 & 1 & | & 2 \end{pmatrix}$$

$$\begin{array}{l} R_2 \leftrightarrow R_3 \\ \frac{1}{7}R_2 \rightarrow R_2 \end{array} \xrightarrow{\hspace{1cm}} \begin{pmatrix} 1 & -1 & 0 & | & 2 \\ 0 & 7 & 0 & | & -7 \\ 0 & 3 & 1 & | & -1 \\ 0 & 0 & 1 & | & 2 \end{pmatrix} \xrightarrow{\hspace{1cm}} \begin{pmatrix} 1 & -1 & 0 & | & 2 \\ 0 & 1 & 0 & | & -1 \\ 0 & 3 & 1 & | & -1 \\ 0 & 0 & 1 & | & 2 \end{pmatrix}$$

$$-3R_2 + R_3 \rightarrow R_3 \xrightarrow{\hspace{1cm}} \begin{pmatrix} 1 & -1 & 0 & | & 2 \\ 0 & 1 & 0 & | & -1 \\ 0 & 0 & 1 & | & 2 \\ 0 & 0 & 1 & | & 2 \end{pmatrix}$$

$$-R_3 + R_4 \rightarrow R_4 \xrightarrow{\hspace{1cm}} \begin{pmatrix} 1 & -1 & 0 & | & 2 \\ 0 & 1 & 0 & | & -1 \\ 0 & 0 & 1 & | & 2 \\ 0 & 0 & 0 & | & 0 \end{pmatrix} \xrightarrow{\hspace{1cm}}$$

The reduced system is

$$\begin{array}{rcl} c_1 - c_2 & = & 2 \quad (1) \\ c_2 & = & -1 \quad (2) \\ c_3 & = & 2 \quad (3) \\ 0 & = & 0 \end{array}$$

We get the solution

$$\begin{array}{l} c_3 = 2 \\ c_2 = -1 \\ c_1 = 2 + c_2 = 2 - 1 = 1 \end{array}$$

Thus,

$$3 + 2x + x^2 + 2x^3 = 1 \cdot \vec{P}_1 - 1 \cdot \vec{P}_2 + 2 \cdot \vec{P}_3$$

So,

$3 + 2x + x^2 + 2x^3$  is in the span of  $\vec{P}_1, \vec{P}_2, \vec{P}_3$

③(b)

We want to know if we can write

$$1 + x = c_1 \vec{p}_1 + c_2 \vec{p}_2 + c_3 \vec{p}_3$$

$$1 + 1 \cdot x + 0 \cdot x^2 + 0 \cdot x^3 = c_1(2 + x + 4x^2) + c_2(1 - x + 3x^2) + c_3(1 + x^3)$$

This simplifies to

$$1 + 1 \cdot x + 0 \cdot x^2 + 0 \cdot x^3 = (2c_1 + c_2 + c_3) + (c_1 - c_2)x + (4c_1 + 3c_2)x^2 + c_3x^3$$

Now equate the coefficients of both sides to get:

$$1 = 2c_1 + c_2 + c_3$$

$$1 = c_1 - c_2$$

$$0 = 4c_1 + 3c_2$$

$$0 = c_3$$

Now we must see if this system has a solution or not



$$\left( \begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 & 1 & 0 \\ 4 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array} \right) \xrightarrow{R_1 \leftrightarrow R_2} \left( \begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 1 & 0 \\ 2 & 1 & 1 & 1 & 1 & 0 \\ 4 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array} \right)$$

$$\begin{array}{l} -2R_1 + R_2 \rightarrow R_2 \\ -4R_1 + R_3 \rightarrow R_3 \end{array} \xrightarrow{\hspace{1cm}} \left( \begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 1 & 0 \\ 0 & 3 & 1 & -1 & -1 & 0 \\ 0 & 7 & 0 & -4 & -4 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array} \right)$$

$$\begin{array}{l} R_2 \leftrightarrow R_3 \\ \frac{1}{7}R_2 \rightarrow R_2 \end{array} \xrightarrow{\hspace{1cm}} \left( \begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 1 & 0 \\ 0 & 7 & 0 & -4 & -4 & 0 \\ 0 & 3 & 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array} \right) \xrightarrow{\hspace{1cm}} \left( \begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -4/7 & -4/7 & 0 \\ 0 & 3 & 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array} \right)$$

$$-3R_2 + R_3 \rightarrow R_3 \xrightarrow{\hspace{1cm}} \left( \begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -4/7 & -4/7 & 0 \\ 0 & 0 & 1 & 5/7 & 5/7 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array} \right)$$

$$-R_3 + R_4 \rightarrow R_4 \xrightarrow{\hspace{1cm}} \left( \begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -4/7 & -4/7 & 0 \\ 0 & 0 & 1 & 5/7 & 5/7 & 0 \\ 0 & 0 & 0 & -5/7 & -5/7 & 0 \end{array} \right) \xrightarrow{\hspace{1cm}}$$

The reduced system is

$$\begin{array}{rcl} c_1 - c_2 & = & 1 \quad (1) \\ c_2 & = & -4/7 \quad (2) \\ c_3 & = & 5/7 \quad (3) \\ 0 & = & -5/7 \quad (4) \end{array}$$

Equation (4) is  $0 = -5/7$ .

Which isn't true.

Hence the system has no solutions.

Thus,

$$3 + 2x + x^2 + 2x^3 = c_1 \cdot \vec{P}_1 + c_2 \cdot \vec{P}_2 + c_3 \cdot \vec{P}_3$$

Cannot be solved for  $c_1, c_2, c_3$ .

So,

$3 + 2x + x^2 + 2x^3$  is not in the span of  $\vec{P}_1, \vec{P}_2, \vec{P}_3$ .

③(c)

We have that

$$0 = 0 \cdot \vec{p}_1 + 0 \cdot \vec{p}_2 + 0 \vec{p}_3$$

Thus,  $0$  is in the span of  
 $\vec{p}_1, \vec{p}_2, \vec{p}_3$ .

③ (d)

We want to know if we can write

$$4 - x + 10x^2 = c_1 \vec{p}_1 + c_2 \vec{p}_2 + c_3 \vec{p}_3$$

$$4 - x + 10x^2 = c_1(2 + x + 4x^2) + c_2(1 - x + 3x^2) + c_3(1 + x^3)$$

This simplifies to

$$4 - x + 10x^2 + 0x^3 = (2c_1 + c_2 + c_3) + (c_1 - c_2)x + (4c_1 + 3c_2)x^2 + c_3x^3$$

Now equate the coefficients of both sides to get:

$$\begin{aligned} 4 &= 2c_1 + c_2 + c_3 \\ -1 &= c_1 - c_2 \\ 10 &= 4c_1 + 3c_2 \\ 0 &= c_3 \end{aligned}$$

Now we must see if this system has a solution or not

$$\begin{pmatrix} 2 & 1 & 1 & 4 \\ 1 & -1 & 0 & -1 \\ 4 & 3 & 0 & 10 \\ 0 & 0 & 1 & 0 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & -1 & 0 & -1 \\ 2 & 1 & 1 & 4 \\ 4 & 3 & 0 & 10 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\begin{array}{l} -2R_1 + R_2 \rightarrow R_2 \\ -4R_1 + R_3 \rightarrow R_3 \end{array} \xrightarrow{\hspace{1cm}} \begin{pmatrix} 1 & -1 & 0 & -1 \\ 0 & 3 & 1 & 6 \\ 0 & 7 & 0 & 14 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\begin{array}{l} R_2 \leftrightarrow R_3 \\ \frac{1}{7}R_2 \rightarrow R_2 \end{array} \xrightarrow{\hspace{1cm}} \begin{pmatrix} 1 & -1 & 0 & -1 \\ 0 & 7 & 0 & 14 \\ 0 & 3 & 1 & 6 \\ 0 & 0 & 1 & 0 \end{pmatrix} \xrightarrow{\hspace{1cm}} \begin{pmatrix} 1 & -1 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 3 & 1 & 6 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$-3R_2 + R_3 \rightarrow R_3 \xrightarrow{\hspace{1cm}} \begin{pmatrix} 1 & -1 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$-R_3 + R_4 \rightarrow R_4 \xrightarrow{\hspace{1cm}} \begin{pmatrix} 1 & -1 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{\hspace{2cm}}$$

The reduced system is

$$\begin{array}{rcl} c_1 - c_2 & = & -1 \quad (1) \\ c_2 & = & 2 \quad (2) \\ c_3 & = & 0 \quad (3) \\ 0 & = & 0 \end{array}$$

We get the solution

$$\begin{array}{l} c_3 = 0 \\ c_2 = 2 \\ c_1 = -1 + c_2 = -1 + 2 = 1 \end{array}$$

Thus,

$$4 - x + 10x^2 = 1 \cdot \vec{P}_1 + 2 \cdot \vec{P}_2 + 0 \cdot \vec{P}_3$$

So,

$4 - x + 10x^2$  is in the span  
of  $\vec{P}_1, \vec{P}_2, \vec{P}_3$

(4) (a) We must solve the equation

$$c_1 \vec{u}_1 + c_2 \vec{u}_2 = \vec{0}$$

This is equivalent to

$$c_1 \langle 1, -1 \rangle + c_2 \langle 2, 1 \rangle = \langle 0, 0 \rangle$$

This becomes

$$\langle c_1, -c_1 \rangle + \langle 2c_2, c_2 \rangle = \langle 0, 0 \rangle$$

which is

$$\langle c_1 + 2c_2, -c_1 + c_2 \rangle = \langle 0, 0 \rangle$$

This becomes

$$\begin{cases} c_1 + 2c_2 = 0 \\ -c_1 + c_2 = 0 \end{cases}$$

Solving we get:

$$\left( \begin{array}{cc|c} 1 & 2 & 0 \\ -1 & 1 & 0 \end{array} \right) \xrightarrow{R_1 + R_2 \rightarrow R_2} \left( \begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 3 & 0 \end{array} \right) \xrightarrow{\frac{1}{3}R_2 \rightarrow R_2} \left( \begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & 0 \end{array} \right)$$

Thus,  $\begin{cases} c_1 + 2c_2 = 0 & \textcircled{1} \\ c_2 = 0 & \textcircled{2} \end{cases} \Rightarrow \begin{cases} \textcircled{2} c_2 = 0 \\ \textcircled{1} c_1 = -2c_2 = -2(0) = 0 \end{cases}$

Since the only solution to  $c_1 \vec{u}_1 + c_2 \vec{u}_2 = \vec{0}$  is  $c_1 = 0, c_2 = 0$  we know that  $\vec{u}_1 = \langle 1, -1 \rangle$ ,  $\vec{u}_2 = \langle 2, 1 \rangle$  are linearly independent.

4(b) Method 1 - the long way

We must find the solutions to the equation

$$c_1 \vec{u}_1 + c_2 \vec{u}_2 + c_3 \vec{u}_3 = \vec{0}$$

This equation becomes

$$c_1 \langle 3, -1 \rangle + c_2 \langle 4, 5 \rangle + c_3 \langle -4, 7 \rangle = \langle 0, 0 \rangle$$

This is equivalent to

$$\langle 3c_1, -c_1 \rangle + \langle 4c_2, 5c_2 \rangle + \langle -4c_3, 7c_3 \rangle = \langle 0, 0 \rangle$$

This is equivalent to

$$\langle 3c_1 + 4c_2 - 4c_3, -c_1 + 5c_2 + 7c_3 \rangle = \langle 0, 0 \rangle$$

Thus,

$$\begin{aligned} 3c_1 + 4c_2 - 4c_3 &= 0 \\ -c_1 + 5c_2 + 7c_3 &= 0 \end{aligned}$$

Let's solve this system.

$$\left( \begin{array}{ccc|c} 3 & 4 & -4 & 0 \\ -1 & 5 & 7 & 0 \end{array} \right) \xrightarrow{R_1 \leftrightarrow R_2} \left( \begin{array}{ccc|c} -1 & 5 & 7 & 0 \\ 3 & 4 & -4 & 0 \end{array} \right) \rightarrow$$



$$\xrightarrow{-R_1 \rightarrow R_1} \left( \begin{array}{ccc|c} 1 & -5 & -7 & 0 \\ 3 & 4 & -4 & 0 \end{array} \right)$$

$$\xrightarrow{-3R_1 + R_2 \rightarrow R_2} \left( \begin{array}{ccc|c} 1 & -5 & -7 & 0 \\ 0 & 19 & 17 & 0 \end{array} \right)$$

$$\xrightarrow{\frac{1}{19}R_2 \rightarrow R_2} \left( \begin{array}{ccc|c} 1 & -5 & -7 & 0 \\ 0 & 1 & \frac{17}{19} & 0 \end{array} \right)$$

So, we get:

$$\begin{cases} c_1 - 5c_2 - 7c_3 = 0 \\ c_2 + \frac{17}{19}c_3 = 0 \end{cases}$$



$$\begin{cases} c_1 = 5c_2 + 7c_3 \\ c_2 = -\frac{17}{19}c_3 \end{cases} \Rightarrow$$

Leading variables:  $c_1, c_2$

Free variables:  $c_3$

$$\begin{aligned} c_3 &= t \\ c_2 &= -\frac{17}{19}t \\ c_1 &= 5c_2 + 7c_3 \\ &= -\frac{85}{19}t + 7t \\ &= \frac{48}{19}t \end{aligned} \rightarrow$$

So,

$$c_1 \vec{u}_1 + c_2 \vec{u}_2 + c_3 \vec{u}_3 = \vec{0}$$

can be solved by

$$\left(\frac{48}{19}t\right) \cdot \vec{u}_1 - \left(\frac{17}{19}t\right) \cdot \vec{u}_2 + t \vec{u}_3 = \vec{0}$$

for any  $t$ .

In particular, say we set  $t=19$ ,

Then we get:

$$48 \vec{u}_1 - 17 \vec{u}_2 + 19 \vec{u}_3 = \vec{0}.$$

Thus,  $\vec{u}_1, \vec{u}_2, \vec{u}_3$  are linearly dependent.

Method 2 -  
short way

The dimension of  $\mathbb{R}^2$  is 2. Thus, if we have more than 2 vectors in  $\mathbb{R}^2$  they must be linearly dependent by a theorem in class. Since we have 3 vectors in a 2-dimensional space,  $\vec{u}_1, \vec{u}_2, \vec{u}_3$  are lin. dep.

④ (c)

Consider the equation

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 = \vec{0}.$$

The above equation becomes

$$c_1 \langle -3, 0, 4 \rangle + c_2 \langle 5, -1, 2 \rangle + c_3 \langle 1, 1, 3 \rangle = \langle 0, 0, 0 \rangle$$

which is equivalent to

$$\langle -3c_1, 0, 4c_1 \rangle + \langle 5c_2, -c_2, 2c_2 \rangle + \langle c_3, c_3, 3c_3 \rangle = \langle 0, 0, 0 \rangle$$

which becomes

$$\langle -3c_1 + 5c_2 + c_3, -c_2 + c_3, 4c_1 + 2c_2 + 3c_3 \rangle = \langle 0, 0, 0 \rangle$$

which is equivalent to

$$\begin{aligned} -3c_1 + 5c_2 + c_3 &= 0 \\ -c_2 + c_3 &= 0 \\ 4c_1 + 2c_2 + 3c_3 &= 0 \end{aligned}$$

Let's solve this system.

$$\begin{pmatrix} -3 & 5 & 1 & | & 0 \\ 0 & -1 & 1 & | & 0 \\ 4 & 2 & 3 & | & 0 \end{pmatrix} \xrightarrow{-\frac{1}{3}R_1 \rightarrow R_1} \begin{pmatrix} 1 & -\frac{5}{3} & -\frac{1}{3} & | & 0 \\ 0 & -1 & 1 & | & 0 \\ 4 & 2 & 3 & | & 0 \end{pmatrix}$$

$$\xrightarrow{-4R_1 + R_3 \rightarrow R_3} \begin{pmatrix} 1 & -\frac{5}{3} & -\frac{1}{3} & | & 0 \\ 0 & -1 & 1 & | & 0 \\ 0 & \frac{26}{3} & \frac{13}{3} & | & 0 \end{pmatrix}$$

$$\begin{matrix} -R_2 \rightarrow R_2 \\ 3R_3 \rightarrow R_3 \end{matrix} \xrightarrow{\hspace{1cm}} \begin{pmatrix} 1 & -\frac{5}{3} & -\frac{1}{3} & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 0 & 26 & 13 & | & 0 \end{pmatrix}$$

$$\xrightarrow{-26R_2 + R_3 \rightarrow R_3} \begin{pmatrix} 1 & -\frac{5}{3} & -\frac{1}{3} & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 0 & 0 & 39 & | & 0 \end{pmatrix}$$

$$\xrightarrow{\frac{1}{39}R_3 \rightarrow R_3} \begin{pmatrix} 1 & -\frac{5}{3} & -\frac{1}{3} & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{pmatrix}$$

This becomes

$$\begin{aligned} c_1 - \frac{5}{3}c_2 - \frac{1}{3}c_3 &= 0 & \textcircled{1} \\ c_2 - c_3 &= 0 & \textcircled{2} \\ c_3 &= 0 & \textcircled{3} \end{aligned}$$

$$\textcircled{3} \text{ gives } c_3 = 0$$

$$\textcircled{2} \text{ gives } c_2 = c_3 = 0$$

$$\textcircled{1} \text{ gives } c_1 = \frac{5}{3}c_2 + \frac{1}{3}c_3 = \frac{5}{3} \cdot 0 + \frac{1}{3} \cdot 0 = 0$$

Thus, the only solution to  
 $c_1 \vec{u}_1 + c_2 \vec{u}_2 + c_3 \vec{u}_3 = \vec{0}$

$$\text{is } c_1 = c_2 = c_3 = 0.$$

So,  $\vec{u}_1, \vec{u}_2, \vec{u}_3$  are linearly independent.

(4) (d)

Consider the equation

$$c_1 \vec{p}_1 + c_2 \vec{p}_2 + c_3 \vec{p}_3 = \vec{0}.$$

If the only solution to this equation is  $c_1 = c_2 = c_3 = 0$ , then  $\vec{p}_1, \vec{p}_2, \vec{p}_3$  are linearly independent. If there are more solutions then  $\vec{p}_1, \vec{p}_2, \vec{p}_3$  are linearly dependent. Let's see what happens.

The above equation becomes

$$c_1(3 - 2x + x^2) + c_2(1 + x + x^2) + c_3(6 - 4x + 2x^2) = 0 + 0x + 0x^2$$

Grouping like terms gives

$$(3c_1 + c_2 + 6c_3) + (-2c_1 + c_2 - 4c_3)x + (c_1 + c_2 + 2c_3) = 0 + 0x + 0x^2$$

Equating coefficients gives

$$\begin{aligned} 3c_1 + c_2 + 6c_3 &= 0 \\ -2c_1 + c_2 - 4c_3 &= 0 \\ c_1 + c_2 + 2c_3 &= 0 \end{aligned}$$

Let's solve this system.

$$\left( \begin{array}{ccc|c} 3 & 1 & 6 & 0 \\ -2 & 1 & -4 & 0 \\ 1 & 1 & 2 & 0 \end{array} \right) \xrightarrow{R_1 \leftrightarrow R_3} \left( \begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ -2 & 1 & -4 & 0 \\ 3 & 1 & 6 & 0 \end{array} \right)$$

$$\begin{array}{l} 2R_1 + R_2 \rightarrow R_2 \\ -3R_1 + R_3 \rightarrow R_3 \end{array} \rightarrow \left( \begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{array} \right)$$

$$\begin{array}{l} \frac{1}{3}R_2 \rightarrow R_2 \\ -\frac{1}{2}R_3 \rightarrow R_3 \end{array} \rightarrow \left( \begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right)$$

$$-R_2 + R_3 \rightarrow R_3 \rightarrow \left( \begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

The reduced system is:

$$\begin{array}{rcl} c_1 + c_2 + 2c_3 & = & 0 \\ c_2 & = & 0 \\ 0 & = & 0 \end{array}$$

Leading variables  
are  $c_1, c_2$ .  
Free variable  
is  $c_3$

$$c_1 = -c_2 - 2c_3 \quad (1)$$

$$c_2 = 0 \quad (2)$$



$$c_3 = t \quad (\text{set free variable})$$

$$(2) \quad c_2 = 0$$

$$(1) \quad c_1 = -c_2 - 2c_3 \\ = 0 - 2t = -2t$$

So,

$$(-2t) \vec{p}_1 + 0 \cdot \vec{p}_2 + t \cdot \vec{p}_3 = \vec{0}$$

for every  $t$ .

In particular, for  $t=1$  we get

$$-2 \vec{p}_1 + 0 \vec{p}_2 + 1 \cdot \vec{p}_3 = \vec{0}.$$

Thus,  $\vec{p}_1, \vec{p}_2, \vec{p}_3$  are linearly

dependent.



(4) (e)

Consider the equation

$$c_1 \vec{p}_1 + c_2 \vec{p}_2 + c_3 \vec{p}_3 = \vec{0}$$

which becomes

$$c_1 (1) + c_2 (1+x) + c_3 (1+x+x^2) = \underbrace{0+0x+0x^2}_{\vec{0}}$$

Regrouping we get

$$\underbrace{(c_1 + c_2 + c_3)} + \underbrace{(c_2 + c_3)}x + \underbrace{c_3}x^2 = \underbrace{0+0x+0x^2}$$

Equating coefficients we get :

$c_1 + c_2 + c_3 = 0$	①	} already reduced
$c_2 + c_3 = 0$	②	
$c_3 = 0$	③	

Solving we get

③  $c_3 = 0$ , ②  $c_2 = -c_3 = -(0) = 0$ , ①  $c_1 = -c_2 - c_3 = -(0) - (0) = 0$

Since the only solution to  $c_1 \vec{p}_1 + c_2 \vec{p}_2 + c_3 \vec{p}_3 = \vec{0}$

is  $c_1 = 0, c_2 = 0, c_3 = 0$  we know that

$\vec{p}_1 = 1, \vec{p}_2 = 1+x, \vec{p}_3 = 1+x+x^2$  are linearly independent.

⑤ (a)

We want to check whether or not the vectors  $\vec{v}_1 = \langle 2, 2, 2 \rangle$ ,  $\vec{v}_2 = \langle 4, 1, 2 \rangle$ ,  $\vec{v}_3 = \langle 0, 1, 1 \rangle$  are linearly independent or linearly dependent in  $\mathbb{R}^3$ .

We want to solve  $c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 = \vec{0}$

for  $c_1, c_2, c_3$ .

Suppose,

$$c_1 \langle 2, 2, 2 \rangle + c_2 \langle 4, 1, 2 \rangle + c_3 \langle 0, 1, 1 \rangle = \langle 0, 0, 0 \rangle$$

Then,

$$\langle 2c_1 + 4c_2, 2c_1 + c_2 + c_3, 2c_1 + 2c_2 + c_3 \rangle = \langle 0, 0, 0 \rangle$$

which becomes

$$\begin{cases} 2c_1 + 4c_2 = 0 \\ 2c_1 + c_2 + c_3 = 0 \\ 2c_1 + 2c_2 + c_3 = 0 \end{cases}$$



Let's try to solve this system



$$\left( \begin{array}{ccc|c} 2 & 4 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 2 & 2 & 1 & 0 \end{array} \right) \xrightarrow{\frac{1}{2}R_1 \rightarrow R_1} \left( \begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 2 & 2 & 1 & 0 \end{array} \right)$$

$$\begin{array}{l} -2R_1 + R_2 \rightarrow R_2 \\ -2R_1 + R_3 \rightarrow R_3 \end{array} \xrightarrow{\hspace{1cm}} \left( \begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & -2 & 1 & 0 \end{array} \right)$$

$$\xrightarrow{-\frac{1}{3}R_2 \rightarrow R_2} \left( \begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & -\frac{1}{3} & 0 \\ 0 & -2 & 1 & 0 \end{array} \right)$$

$$\xrightarrow{2R_2 + R_3 \rightarrow R_3} \left( \begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & 0 \end{array} \right) \xrightarrow{3R_3 \rightarrow R_3} \left( \begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 \end{array} \right)$$

The reduced system is

$$c_1 + 2c_2 = 0$$

$$c_2 - \frac{1}{3}c_3 = 0$$

$$c_3 = 0$$

which gives  $c_3 = 0$

$$c_2 = \frac{1}{3}c_3 = \frac{1}{3} \cdot 0 = 0$$

$$c_1 = -2c_2 = -2 \cdot 0 = 0.$$

Therefore,  $\vec{v}_1, \vec{v}_2, \vec{v}_3$  are linearly independent in  $\mathbb{R}^3$ .

⑤ (b) Same method as in 4(a).

Suppose

$$c_1 \langle 2, -1, 3 \rangle + c_2 \langle 4, 1, 2 \rangle + c_3 \langle 8, -1, 8 \rangle = \langle 0, 0, 0 \rangle$$

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 = \vec{0}$$

We want to solve for  $c_1, c_2, c_3$ .

This equation becomes

$$\langle 2c_1 - c_1, 3c_1 \rangle + \langle 4c_2, c_2, 2c_2 \rangle + \langle 8c_3, -c_3, 8c_3 \rangle = \langle 0, 0, 0 \rangle$$

This becomes

$$\langle 2c_1 + 4c_2 + 8c_3, -c_1 + c_2 - c_3, 3c_1 + 2c_2 + 8c_3 \rangle = \langle 0, 0, 0 \rangle$$

which gives us the system:

$$\begin{cases} 2c_1 + 4c_2 + 8c_3 = 0 \\ -c_1 + c_2 - c_3 = 0 \\ 3c_1 + 2c_2 + 8c_3 = 0 \end{cases}$$

Let's solve the system

$$\left( \begin{array}{ccc|c} 2 & 4 & 8 & 0 \\ -1 & 1 & -1 & 0 \\ 3 & 2 & 8 & 0 \end{array} \right) \xrightarrow{\frac{1}{2}R_1 \rightarrow R_1} \left( \begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ -1 & 1 & -1 & 0 \\ 3 & 2 & 8 & 0 \end{array} \right)$$

$$\begin{array}{l} R_1 + R_2 \rightarrow R_2 \\ -3R_1 + R_3 \rightarrow R_3 \end{array} \rightarrow \left( \begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & -4 & -4 & 0 \end{array} \right) \rightarrow$$

$$\frac{1}{3}R_2 \rightarrow R_2 \rightarrow \left( \begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -4 & -4 & 0 \end{array} \right)$$

$$4R_2 + R_3 \rightarrow R_3 \rightarrow \left( \begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

The reduced system is :

$$\begin{aligned} c_1 + 2c_2 + 4c_3 &= 0 \\ c_2 + c_3 &= 0 \\ 0 &= 0 \end{aligned}$$

leading variables:  $c_1, c_2$   
free variable:  $c_3$

This gives:

$$\begin{aligned} c_1 &= -2c_2 - 4c_3 & \textcircled{1} \\ c_2 &= -c_3 & \textcircled{2} \\ c_3 &= t & \textcircled{3} \end{aligned}$$

Thus,

$$\textcircled{3} \quad c_3 = t$$

$$\textcircled{2} \quad c_2 = -c_3 = -t$$

$$\textcircled{1} \quad c_1 = -2c_2 - 4c_3 = -2(-t) - 4t = -2t$$

So, the solution is:

$$\begin{aligned} c_1 &= -2t \\ c_2 &= -t \\ c_3 &= t \end{aligned} \quad t \text{ can be any real \#}$$

For example,  $t=1$  gives  $c_1=-2, c_2=-1, c_3=1$ .

Thus, plugging into the original equation gives

$$-2 \cdot \langle 2, -1, 3 \rangle - 1 \cdot \langle 4, 1, 2 \rangle + 1 \cdot \langle 8, -1, 8 \rangle = \langle 0, 0, 0 \rangle$$

$$(-2) \cdot \vec{v}_1 - 1 \cdot \vec{v}_2 + 1 \cdot \vec{v}_3 = \vec{0}$$

Thus,  $\vec{v}_1 = \langle 2, -1, 3 \rangle$ ,  $\vec{v}_2 = \langle 4, 1, 2 \rangle$ ,  $\vec{v}_3 = \langle 8, -1, 8 \rangle$  are linearly dependent.

⑥(a) In problem 5(a) we showed that

$$\vec{v}_1 = \langle 2, 2, 2 \rangle, \vec{v}_2 = \langle 4, 1, 2 \rangle, \vec{v}_3 = \langle 0, 1, 1 \rangle$$

are linearly independent.

Since we have 3 linearly independent vectors in a 3-dimensional space  $V = \mathbb{R}^3$ ,

by a theorem in class we know

$\vec{v}_1, \vec{v}_2, \vec{v}_3$  must span  $\mathbb{R}^3$  and thus

be a basis for  $V = \mathbb{R}^3$ .

---

⑥(b) In problem 5(b) we showed that

$$\vec{v}_1 = \langle 2, -1, 3 \rangle, \vec{v}_2 = \langle 4, 1, 2 \rangle, \vec{v}_3 = \langle 8, -1, 8 \rangle$$

are linearly dependent.

Thus,  $\vec{v}_1, \vec{v}_2, \vec{v}_3$  are not a basis

for  $V = \mathbb{R}^3$ .

(7) (a) Since  $\mathbb{R}^3$  has dimension 3, we need 3 vectors to have a basis for  $\mathbb{R}^3$ . Thus,  $\vec{v}_1 = \langle 4, -1, 2 \rangle$ ,  $\vec{v}_2 = \langle -4, 10, 2 \rangle$  are not a basis for  $\mathbb{R}^3$ .

---

(7) (b) By problem 4(c) the vectors  $\vec{v}_1 = \langle -3, 0, 4 \rangle$ ,  $\vec{v}_2 = \langle 5, -1, 2 \rangle$ ,  $\vec{v}_3 = \langle 1, 1, 3 \rangle$  are linearly independent.

Since we have 3 linearly independent vectors  $\vec{v}_1, \vec{v}_2, \vec{v}_3$  in a vector space  $\mathbb{R}^3$  of dimension 3, by a theorem in class they must span  $\mathbb{R}^3$  and hence are a basis for  $\mathbb{R}^3$ .



⑦ (c) Since  $\mathbb{R}^3$  has dimension 3, we need exactly 3 vectors to have a basis for  $\mathbb{R}^3$ . Thus,  $\vec{v}_1 = \langle -2, 0, 1 \rangle$ ,  $\vec{v}_2 = \langle 3, 2, 5 \rangle$ ,  $\vec{v}_3 = \langle 6, -1, 1 \rangle$ ,  $\vec{v}_4 = \langle 7, 0, -2 \rangle$  are not a basis for  $\mathbb{R}^3$ . We have too many vectors.

You could also just directly show that these 4 vectors are linearly dependent and hence not a basis for  $\mathbb{R}^3$ .

⑧(a)

The dimension of  $P_2$  is  $2+1=3$ ,

In problem 4(e) we showed that

$$\vec{p}_1 = 1, \vec{p}_2 = 1+x, \vec{p}_3 = 1+x+x^2$$

are linearly independent.

Thus, by a theorem in class,

since we have 3 linearly

independent vectors in a 3-dimensional

space  $V = P_2$ , we know that

$$\vec{p}_1 = 1, \vec{p}_2 = 1+x, \vec{p}_3 = 1+x+x^2$$

form a basis for  $V = P_2$ .

⑧ (b) Same idea as 8(a).

The dimension of  $P_2$  is  $2+1=3$ .

Thus, since we have 3 vectors, the vectors

$$\vec{p}_1 = 6 - x^2, \quad \vec{p}_2 = 1 + x + 4x^2, \quad \vec{p}_3 = 8 + 2x + 7x^2$$

will be a basis if and only if they are linearly independent.

Consider the equation

$$c_1 \vec{p}_1 + c_2 \vec{p}_2 + c_3 \vec{p}_3 = \vec{0}$$

which becomes

$$c_1(6 - x^2) + c_2(1 + x + 4x^2) + c_3(8 + 2x + 7x^2) = 0 + 0x + 0x^2$$

Grouping like terms gives

$$(6c_1 + c_2 + 8c_3) + (c_2 + 2c_3)x + (-c_1 + 4c_2 + 7c_3) = 0 + 0x + 0x^2$$

Equating coefficients gives

$$\begin{cases} 6c_1 + c_2 + 8c_3 = 0 \\ c_2 + 2c_3 = 0 \\ -c_1 + 4c_2 + 7c_3 = 0 \end{cases}$$

Let's solve this system

$$\left( \begin{array}{ccc|c} 6 & 1 & 8 & 0 \\ 0 & 1 & 2 & 0 \\ -1 & 4 & 7 & 0 \end{array} \right) \xrightarrow{R_1 \leftrightarrow R_3} \left( \begin{array}{ccc|c} -1 & 4 & 7 & 0 \\ 0 & 1 & 2 & 0 \\ 6 & 1 & 8 & 0 \end{array} \right)$$

$$\xrightarrow{-R_1 \rightarrow R_1} \left( \begin{array}{ccc|c} 1 & -4 & -7 & 0 \\ 0 & 1 & 2 & 0 \\ 6 & 1 & 8 & 0 \end{array} \right)$$

$$\xrightarrow{-6R_1 + R_3 \rightarrow R_3} \left( \begin{array}{ccc|c} 1 & -4 & -7 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 25 & 50 & 0 \end{array} \right)$$

$$\xrightarrow{-25R_2 + R_3 \rightarrow R_3} \left( \begin{array}{ccc|c} 1 & -4 & -7 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

The reduced system is :

$$\begin{cases} c_1 - 4c_2 - 7c_3 = 0 \\ c_2 + 2c_3 = 0 \\ 0 = 0 \end{cases}$$

Leading variables:  $c_1, c_2$

Free variable:  $c_3$

Solutions:

$$c_3 = t$$

$$c_2 = -2c_3 = -2t$$

$$c_1 = 4c_2 + 7c_3 = 4(-2t) + 7t = -t$$

Therefore,

$$(-t) \vec{p}_1 + (-2t) \vec{p}_2 + (t) \vec{p}_3 = \vec{0}$$

for any  $t$ .

For example, if  $t=1$ , then

$$-\vec{p}_1 - 2\vec{p}_2 + \vec{p}_3 = \vec{0}$$

Thus,  $\vec{p}_1, \vec{p}_2, \vec{p}_3$  are linearly dependent

and hence are not a basis

for  $P_2$ .

9(a)

Let's show that the vectors  $\langle 1, 4 \rangle, \langle 3, -2 \rangle$  are linearly independent.

Consider the equation

$$c_1 \langle 1, 4 \rangle + c_2 \langle 3, -2 \rangle = \langle 0, 0 \rangle.$$

This becomes

$$\langle c_1 + 3c_2, 4c_1 - 2c_2 \rangle = \langle 0, 0 \rangle$$

which is equivalent to

$$\begin{cases} c_1 + 3c_2 = 0 \\ 4c_1 - 2c_2 = 0 \end{cases}$$

Let's solve this.

$$\left( \begin{array}{cc|c} 1 & 3 & 0 \\ 4 & -2 & 0 \end{array} \right) \xrightarrow{-4R_1 + R_2 \rightarrow R_2} \left( \begin{array}{cc|c} 1 & 3 & 0 \\ 0 & -14 & 0 \end{array} \right)$$

$$\xrightarrow{-\frac{1}{14}R_2 \rightarrow R_2} \left( \begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 1 & 0 \end{array} \right).$$

So, the reduced system is

$$\begin{cases} c_1 + 3c_2 = 0 \\ c_2 = 0 \end{cases}$$



which gives

$$c_2 = 0, c_1 = -3c_2 = 0.$$

Thus, the only solution is  $c_1 = c_2 = 0$ .

Thus,  $\langle 1, 4 \rangle, \langle 3, -2 \rangle$  are linearly independent.

Since  $\mathbb{R}^2$  has dimension 2, and we have 2 linearly independent vectors, we can

conclude that  $\langle 1, 4 \rangle, \langle 3, -2 \rangle$  are a basis for  $\mathbb{R}^2$

---

(9)(b) We must solve

$$\langle -7, 14 \rangle = c_1 \langle 1, 4 \rangle + c_2 \langle 3, -2 \rangle$$

which becomes

$$\langle -7, 14 \rangle = \langle c_1 + 3c_2, 4c_1 - 2c_2 \rangle$$

which becomes

$$\begin{cases} -7 = c_1 + 3c_2 \\ 14 = 4c_1 - 2c_2 \end{cases}$$

Let's solve this system:

$$\left( \begin{array}{cc|c} 1 & 3 & -7 \\ 4 & -2 & 14 \end{array} \right) \xrightarrow{-4R_1 + R_2 \rightarrow R_2} \left( \begin{array}{cc|c} 1 & 3 & -7 \\ 0 & -14 & 42 \end{array} \right) \longrightarrow$$

$$\frac{-1}{14}R_2 \rightarrow R_2 \rightarrow \left( \begin{array}{cc|c} 1 & 3 & -7 \\ 0 & 1 & -3 \end{array} \right)$$

So we get:

$$\begin{aligned} c_1 + 3c_2 &= -7 \\ c_2 &= -3 \end{aligned}$$

$$So, c_2 = -3$$

$$c_1 = -7 - 3c_2 = -7 - 3(-3) = 2.$$

Thus,

$$\langle -7, 14 \rangle = 2 \cdot \langle 1, 4 \rangle + (-3) \langle 3, -2 \rangle$$

So, the coordinates of  $\langle -7, 14 \rangle$  with respect to the ordered basis

$$\beta = [\langle 1, 4 \rangle, \langle 3, -2 \rangle] \text{ are}$$

$$[\langle -7, 14 \rangle]_{\beta} = \langle 2, -3 \rangle$$



⑨ (c)

We want to solve

$$\langle 3, -12 \rangle = c_1 \langle 1, 4 \rangle + c_2 \langle 3, -2 \rangle$$

which becomes

$$\langle 3, -12 \rangle = \langle c_1 + 3c_2, 4c_1 - 2c_2 \rangle$$

which is equivalent to

$$\begin{cases} 3 = c_1 + 3c_2 \\ -12 = 4c_1 - 2c_2 \end{cases}$$

Let's solve this system:

$$\left( \begin{array}{cc|c} 1 & 3 & 3 \\ 4 & -2 & -12 \end{array} \right) \xrightarrow{-4R_1 + R_2 \rightarrow R_2} \left( \begin{array}{cc|c} 1 & 3 & 3 \\ 0 & -14 & -24 \end{array} \right)$$

$$\xrightarrow{-\frac{1}{14}R_2 \rightarrow R_2} \left( \begin{array}{cc|c} 1 & 3 & 3 \\ 0 & 1 & \frac{12}{7} \end{array} \right)$$

$$\begin{cases} c_1 + 3c_2 = 3 \\ c_2 = \frac{12}{7} \end{cases}$$

$$\rightarrow c_2 = \frac{12}{7}, \quad c_1 = 3 - 3c_2 = 3 - \frac{36}{7} = -\frac{15}{7}$$

Thus,

$$\langle 3, -12 \rangle = \left(-\frac{15}{7}\right) \cdot \langle 1, 4 \rangle + \left(\frac{12}{7}\right) \cdot \langle 3, -2 \rangle$$

$S_0$ , the coordinates of  $\langle 3, -12 \rangle$  with respect to the ordered basis  $\beta = [\langle 1, 4 \rangle, \langle 3, -2 \rangle]$  are

$$[\langle 3, -12 \rangle]_{\beta} = \left\langle -\frac{15}{7}, \frac{12}{7} \right\rangle$$

10 (a)

Let us show that the vectors are linearly independent.

Consider the equation

$$c_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + c_4 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

This becomes

$$\begin{pmatrix} c_1 + c_2 & c_2 - c_4 \\ c_2 + c_4 & c_1 + c_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Equating coefficients gives

$$\begin{array}{rcl} c_1 + c_2 & & = 0 \\ c_2 & -c_4 & = 0 \\ c_2 & +c_4 & = 0 \\ c_1 & +c_3 & = 0 \end{array}$$

Let's solve this system

$$\left( \begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{array} \right) \xrightarrow{-R_1 + R_4 \rightarrow R_4} \left( \begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 \end{array} \right)$$

$$\begin{array}{l} -R_2 + R_3 \rightarrow R_3 \\ R_2 + R_4 \rightarrow R_4 \end{array} \rightarrow \left( \begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array} \right)$$

$$R_3 \leftrightarrow R_4 \rightarrow \left( \begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{array} \right)$$

$$\frac{1}{2}R_4 \rightarrow R_4 \rightarrow \left( \begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right)$$

This becomes:

$$\begin{array}{rcl} c_1 + c_2 & & = 0 \\ c_2 & - c_4 & = 0 \\ c_3 - c_4 & & = 0 \\ c_4 & & = 0 \end{array}$$

Solving gives  $c_4 = 0$ ,  $c_3 = c_4 = 0$ ,  
 $c_2 = c_4 = 0$ , and  $c_1 = -c_2 = -0 = 0$ ,

Thus,  $c_1 = c_2 = c_3 = c_4 = 0$  is the  
only solution.

So, the vectors

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

are linearly independent.

Since there are 4 of them and the  
dimension of  $M_{2,2}$  is 4,  
they form a basis for  $M_{2,2}$ .

(10)(b) We need to solve

$$\begin{pmatrix} 1 & -2 \\ 0 & -3 \end{pmatrix} = c_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + c_4 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

which becomes

$$\begin{pmatrix} 1 & -2 \\ 0 & -3 \end{pmatrix} = \begin{pmatrix} c_1 + c_2 & c_2 - c_4 \\ c_2 + c_4 & c_1 + c_3 \end{pmatrix}$$

which is equivalent to

$c_1 + c_2$	$= 1$
$c_2$	$-c_4 = -2$
$c_2$	$+c_4 = 0$
$c_1$	$+c_3 = -3$

Let's solve this system

$$\left( \begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & -3 \end{array} \right) \xrightarrow{-R_1 + R_4 \rightarrow R_4} \left( \begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & -4 \end{array} \right)$$

$$\begin{array}{l} -R_2 + R_3 \rightarrow R_3 \\ \hline R_2 + R_4 \rightarrow R_4 \end{array} \left( \begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 1 & -1 & -6 \end{array} \right)$$

$$R_3 \leftrightarrow R_4 \left( \begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & 1 & -1 & -6 \\ 0 & 0 & 0 & 2 & 2 \end{array} \right)$$

$$\frac{1}{2}R_4 \rightarrow R_4 \left( \begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & 1 & -1 & -6 \\ 0 & 0 & 0 & 1 & 1 \end{array} \right)$$

This becomes:

$$\begin{array}{rcl} c_1 + c_2 & & = 1 \\ c_2 & - c_4 & = -2 \\ c_3 - c_4 & & = -6 \\ c_4 & & = 1 \end{array} \rightarrow$$

Solving this system gives

$$c_4 = 1, \quad c_3 = -6 + c_4 = -6 + 1 = -5$$

$$c_2 = -2 + c_4 = -2 + 1 = -1, \quad \text{and}$$

$$c_1 = 1 - c_2 = 1 - (-1) = 2$$

Thus,

$$\begin{pmatrix} 1 & -2 \\ 0 & -3 \end{pmatrix} = 2 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - 1 \cdot \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} - 5 \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

So,

$$\left[ \begin{pmatrix} 1 & -2 \\ 0 & -3 \end{pmatrix} \right]_{\beta} = \langle 2, -1, -5, 1 \rangle$$



(10)(c) We need to solve

$$\begin{pmatrix} 3 & 4 \\ 0 & 1 \end{pmatrix} = c_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + c_4 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

which becomes

$$\begin{pmatrix} 3 & 4 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} c_1 + c_2 & c_2 - c_4 \\ c_2 + c_4 & c_1 + c_3 \end{pmatrix}$$

which is equivalent to

$c_1 + c_2$	$= 3$
$c_2$	$-c_4 = 4$
$c_2$	$+c_4 = 0$
$c_1$	$+c_3 = 1$

Let's solve this system

$$\left( \begin{array}{cccc|c} 1 & 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 & 4 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{array} \right) \xrightarrow{-R_1 + R_4 \rightarrow R_4} \left( \begin{array}{cccc|c} 1 & 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 & 4 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & -2 \end{array} \right)$$

$$\begin{array}{l} -R_2 + R_3 \rightarrow R_3 \\ \hline R_2 + R_4 \rightarrow R_4 \end{array} \left( \begin{array}{cccc|c} 1 & 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 1 & -1 & 2 \end{array} \right)$$

$$R_3 \leftrightarrow R_4 \left( \begin{array}{cccc|c} 1 & 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 2 & -4 \end{array} \right)$$

$$\frac{1}{2}R_4 \rightarrow R_4 \left( \begin{array}{cccc|c} 1 & 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 1 & -2 \end{array} \right)$$

This becomes:

$$\begin{array}{rcl} c_1 + c_2 & & = 3 \\ c_2 & - c_4 & = 4 \\ c_3 - c_4 & & = 2 \\ c_4 & & = -2 \end{array}$$

Solving this system gives

$$c_4 = -2, \quad c_3 = 2 + c_4 = 2 - 2 = 0$$

$$c_2 = 4 + c_4 = 4 - 2 = 2, \quad \text{and}$$

$$c_1 = 3 - c_2 = 3 - 2 = 1,$$

Thus,

$$\begin{pmatrix} 3 & 4 \\ 0 & 1 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 2 \cdot \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - 2 \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

So,

$$\left[ \begin{pmatrix} 3 & 4 \\ 0 & 1 \end{pmatrix} \right]_{\beta} = \langle 1, 2, 0, -2 \rangle$$

$$(11) (a) \text{ Let } \beta = [1, 1+x, 1+x+x^2]$$

We need to solve

$$1-x+2x^2 = c_1(1) + c_2(1+x) + c_3(1+x+x^2)$$

which is

$$1-x+2x^2 = (c_1+c_2+c_3) + (c_2+c_3)x + c_3x^2$$



Equating coefficients gives

$$\begin{cases} 1 = c_1 + c_2 + c_3 & (1) \\ -1 = c_2 + c_3 & (2) \\ 2 = c_3 & (3) \end{cases}$$

} This system is already in reduced form, so we can solve it.

$$\text{We get } c_3 = 2, c_2 = -1 - c_3 = -1 - 2 = -3,$$

$$c_1 = 1 - c_2 - c_3 = 1 - (-3) - 2 = 2.$$

Thus,

$$1-x+2x^2 = 2 \cdot (1) - 3 \cdot (1+x) + 2 \cdot (1+x+x^2)$$

So,

$$[1-x+2x^2]_{\beta} = \langle 2, -3, 2 \rangle$$

$$(11) (b) \text{ Let } \beta = [1, 1+x, 1+x+x^2]$$

We need to solve

$$X = c_1(1) + c_2(1+x) + c_3(1+x+x^2)$$

which is

$$0 + 1 \cdot x + 0 \cdot x^2 = (c_1 + c_2 + c_3) + (c_2 + c_3)x + c_3 x^2$$

Equating coefficients gives

$$\begin{cases} 0 = c_1 + c_2 + c_3 & (1) \\ 1 = c_2 + c_3 & (2) \\ 0 = c_3 & (3) \end{cases}$$

} This system is already in reduced form, so we can solve it.

$$\text{We get } c_3 = 0, c_2 = 1 - c_3 = 1 - 0 = 1$$

$$c_1 = 0 - c_2 - c_3 = 0 - 1 - 0 = -1$$

Thus,

$$X = -1 \cdot (1) + 1 \cdot (1+x) + 0 \cdot (1+x+x^2)$$

So,

$$[X]_{\beta} = \langle -1, 1, 0 \rangle$$

(12)

I claim that

$$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

is a basis for  $M_{2,2}$ .

If we show this claim then  $M_{2,2}$  has dimension 4.

$\beta$  spans  $M_{2,2}$

Let  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  be an arbitrary element of  $M_{2,2}$ .

$$\begin{aligned} \text{Then} \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix} \\ &= a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Thus, every element of  $M_{2,2}$  is in the span of  $\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ .

$\beta$  is linearly independent

Suppose that

$$c_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + c_4 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}_{\vec{0}}$$

Then

$$\begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

So,  $c_1 = c_2 = c_3 = c_4 = 0$ .

Thus,  $\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

is a linearly independent set

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Since  $\beta$  is lin. ind. and spans  $M_{2,2}$ ,

$\beta$  is a basis for  $M_{2,2}$ .

Since  $\beta$  has 4 elements,  $M_{2,2}$  has dimension 4.

(13)

$$P_n = \left\{ a_0 + a_1x + a_2x^2 + \dots + a_nx^n \mid a_0, a_1, \dots, a_n \in \mathbb{R} \right\}$$

Claim:  $\beta = \{1, x, x^2, \dots, x^n\}$  is a basis for  $P_n$ .

β spans  $P_n$ :

Let  $\vec{p} = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$  be an arbitrary element of  $P_n$ . Then

$$\begin{aligned} \vec{p} &= a_0 + a_1x + a_2x^2 + \dots + a_nx^n \\ &= a_0 \cdot 1 + a_1 \cdot x + a_2 \cdot x^2 + \dots + a_n \cdot x^n \end{aligned}$$

So,  $\vec{p}$  is in the span of

$$\beta = \{1, x, x^2, \dots, x^n\}$$



$\beta$  is a linearly independent set of vectors }:

Suppose that

$$c_0 \cdot 1 + c_1 \cdot x + c_2 \cdot x^2 + \dots + c_n \cdot x^n = \underbrace{0 + 0x + 0x^2 + \dots + 0x^n}_{\vec{0}}$$

Then equating coefficients gives

$$c_0 = 0, c_1 = 0, c_2 = 0, \dots, c_n = 0.$$

Thus,  $\beta = \{1, x, x^2, \dots, x^n\}$  is a linearly independent set of vectors.

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From the above,  $\beta = \{1, x, x^2, \dots, x^n\}$  is a basis for  $P_n$ . Since  $\beta$  has  $n+1$  elements in it, the dimension of  $P_n$  is  $n+1$ .